



TITLE:

# Boundedness and Convergence of Solutions of Duffing's Equation (常微分方程式の定性的研究)

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Boundedness and Convergence of Solutions  
of Duffing's Equation

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We shall discuss boundedness of solutions of the equation

$$(1) \quad x'' + f(x)x' + g(x) = e(t) \quad ( ' = d/dt )$$

under suitable conditions. Furthermore, we shall discuss asymptotic stability of a periodic solution and convergence of solutions for the equation

$$(2) \quad x'' + cx' + g(x) = e(t) ,$$

where  $c$  is a positive constant and  $e(t)$  is a periodic function.

This work is motivated by the work of H.Kawakami [2], which gives some numerical computations on the equation

$x'' + kx' + x^3 = B \cos t$  for positive constants  $k$  and  $B$ . There are also very interesting results by C.Hayashi, Y.Ueda and H.Kawakami [1]. Also, our paper heavily depends on the work of W.S.Loud [3].

Theorem 1 In the equation (1) we assume the following conditions (a), (b) and (c).

(a) There exists a solution of (1) under any initial condition.

(b) There exist positive constants  $c$  and  $E$  such that

$$f(x) \geq c \quad \text{and} \quad |e(t)| \leq E .$$

(c)  $g(x)$  is a differentiable function satisfying the following conditions (i), (ii) and (iii).

(i)  $g(x)$  is bounded on any finite interval.

(ii)  $g'(x) \geq 0$

$$(iii) \quad \lim_{x \rightarrow \infty} g(x) > E \text{ and } \lim_{x \rightarrow -\infty} g(x) < -E.$$

By the condition (c),  $g(x)$  is a monotone increasing function, and there exist numbers  $x_1$  and  $x_2$  ( $x_1 < x_2$ ) such that

$$g(x_1) = -E \text{ and } g(x_2) = E.$$

Let  $x(t)$  be any solution of (1). Then there exists a number  $t_0$  such that

$$x_1 - 4E/c^2 \leq x(t) \leq x_2 + 4E/c^2 \text{ and } |x'(t)| \leq 4E/c \text{ for any } t \geq t_0.$$

Our proof is similar to that of Theorem 1 of W.S.Loud [3]. He assumed that  $g'(x) \geq b$  for some positive constant  $b$  in his paper and got an additional information.

Corollary In addition to the conditions (a), (b) and (c) of Theorem 1, we assume the following two conditions,

(d)  $f(x)$  and  $e(t)$  are continuous, and  $f(x)$  satisfies the local Lipschitz condition.

(e)  $e(t)$  is periodic of period  $\tau$  ( $\tau > 0$ ).

Then the equation (1) has a periodic solution of period  $\tau$ .

The equation (2) is a special case of (1), and it is equivalent to the following system of equations.

$$(3) \quad \begin{cases} x' = y \\ y' = -cy - g(x) + e(t) \end{cases}$$

Theorem 2 Assume the following conditions A(i), A(ii) and A(iii).

A(i)  $e(t)$  is a continuous periodic function of period  $\tau$  ( $\tau > 0$ ), and  $E$  is a positive constant such that  $|e(t)| \leq E$ .

A(ii)  $g(x)$  is of class  $C^1$  such that

$g'(x) \geq 0$ ,  $\lim_{x \rightarrow \infty} g(x) > E$ ,  $\lim_{x \rightarrow -\infty} g(x) < -E$ , and  $g'(x) = 0$  only on a countable subset of the real numbers.

A(iii)  $c$  is a positive constant.

Now, let  $n$  be a positive number and let  $x = \phi_1(t)$ ,  $y = \phi_2(t)$  be a non-constant periodic solution of period  $n\tau$  for the equation (3). Suppose that  $|\phi_1(t)| \leq \beta$  for all  $t$  and  $c^2 > H(\beta)$ , where  $H(\beta) = \sup \{g'(x)\}; -\beta \leq x \leq \beta\}$ .

Then the periodic solution  $x = \phi_1(t)$ ,  $y = \phi_2(t)$  is asymptotically stable.

This is a generalization of Loud [3].

Corollary 1 Assume the above conditions A(i), A(ii) and A(iii). Let  $A = \max \{|x_1 - 4E/c^2|, |x_2 + 4E/c^2|\}$  where  $g(x_1) = -E$  and  $g(x_2) = E$ . Further, assume that  $c^2 > H(A) = \sup \{g'(x); -A \leq x \leq A\}$ .

Then every non-constant periodic solution of period  $n\tau$  ( $n$  a positive integer) of the equation (3) is asymptotically stable.

Corollary 2 In addition to the assumptions of Corollary 1, we assume that  $e(t)$  is non-constant.

Then there exists a non-constant periodic solution  $x = \psi_1(t)$ ,  $y = \psi_2(t)$  of period  $\tau$  for the equation (3) such that any periodic solution of period  $n\tau$  (for a suitable positive integer  $n$ ) for the equation (3) coincides with solution  $x = \psi_1(t)$ ,  $y = \psi_2(t)$ .

Theorem 3 Under the same assumption of Corollary 2 of Theorem 2, there exists a unique periodic solution  $x = \psi_1(t)$ ,

$y = \psi_2(t)$  of period  $\tau$  for the equation (3) such that for any solution  $x = x(t)$ ,  $y = y(t)$  of (3) the following equalities hold.

$$\lim_{t \rightarrow \infty} |x(t) - \psi_1(t)| = \lim_{t \rightarrow \infty} |y(t) - \psi_2(t)| = 0$$

This also generalizes the results of Loud [3].

Details will appear elsewhere.

#### References

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